Indian Statistical Institute, Bangalore
M. Math.
First Year, Second Semester
Functional Analysis

Mid-term Examination — Date: 20 February 2024
Maximum marks: 90 — Time: 10:00 AM–1:00 PM (3 hours)
Instructor: Chaitanya G K

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1. Let $X = C^1[a,b]$, the space of all continuously differentiable functions on $[a,b]$. For each $f \in X$, let
$$\|f\| = \|f\|_\infty + \|f'\|_\infty,$$
$$\|f\|_1 = |f(a)| + \|f'\|_\infty,$$
where f' is the derivative of f. Show that X is a Banach space with either of the norms. Show also that, for all $f \in X$,
$$\|f\|_1 \leq \|f\| \leq (b-a+1)\|f\|_1,$$
the constant $b-a+1$ being the best possible. [15]

2. Prove that
$$\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$$
for all $x \in \mathbb{K}^n$. [10]

3. Let X and Y be normed linear spaces such that $X \neq \{0\}$. If $\mathcal{B}(X,Y)$ is a Banach space, prove that Y is a Banach space. [10]

4. Is l^1 reflexive? Justify your answer. [15]

5. Let X be a normed linear space over $\mathbb{C}$ and $f : \mathbb{C} \to X$ be such that $g \circ f$ is bounded and analytic on $\mathbb{C}$ for every $g \in X^*$. Show that f is a constant function on $\mathbb{C}$. [7]

6. State and prove Riesz representation theorem (for functionals on Hilbert spaces). [8]

7. Are the Riesz representation theorem and projection theorem valid in general inner product spaces? Justify your answer. [8+7]

8. Let $T \in \mathcal{B}(H)$, where H is a Hilbert space. Show that T is unitary if and only if it maps every total orthonormal set $\{u_i : i \in \Lambda\}$ onto a total orthonormal set $\{T(u_i) : i \in \Lambda\}$. [10]

9. Let H be a Hilbert space of dimension at least two. Show that there exists a bounded linear operator on H that is not normal. [10]